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A GENERALIZED BOUNDARY VALUE PROBLEM
FOR $u_{xy} = f(x,y,u,u_x,u_y)$

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Mathematics Department Report

A GENERALIZED BOUNDARY VALUE PROBLEM
FOR $u_{xy} = f(x,y,u,u_x,u_y)$

Prepared by:
James Conlan

ABSTRACT: This paper proves an existence theorem for the partial differential equation $u_{xy} = f(x,y,u,u_x,u_y)$ with certain linear combinations of u, u_x and u_y given on two non characteristic curves.

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A Generalized Boundary Value Problem for $u_{xy} = f(x,y,u,u_x,u_y)$

This report deals with a generalization of the Goursat problem for $u_{xy} = f(x,y,u,u_x,u_y)$.

This work was performed under NOL Task No. FR-30.

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Captain, USN
Commander

RICHARD C. ROBERTS
By direction

CONTENTS

A GENERALIZED BOUNDARY VALUE PROBLEM
FOR $u_{xy} = f(x,y,u,u_x,u_y)$

INTRODUCTION

1. The Euler-Cauchy polygon method for proving the existence of a solution for

$$\frac{dy}{dx} = f(x,y) \ , \qquad y(0) = y_0 \ ,$$

was extended by J. B. Diaz [3] to the characteristic boundary value problem for a hyperbolic partial differential equation

(1.1)
$$u_{xy} = f(x,y,u,u_x,u_y) \ ,$$
$$u(x,0) = \sigma(x), \quad u(0,y) = \tau(y) \ , \quad \sigma(0) = \tau(0) \ .$$

Here f is assumed to be bounded and continuous for (x,y) in some rectangle R, and for all possible values of u, u_x, u_y , and to satisfy a Lipschitz condition in u_x, u_y . Under these conditions an existence theorem in the large, i.e., throughout all of R, was obtained.

This method has been further extended by I. I. Glick [4] to the n-dimensional analogue of (1.1),

$$u_{x_1 \ldots x_n} = f \ , \qquad u = \sigma_i \quad \text{on } x_i = 0 \ ,$$

where f is a function of the x_i's of u and of all mixed partial derivatives of u of order less than n, and where the σ_i's satisfy certain obvious compatibility conditions.

In the 2-dimensional case the method was extended by J. Conlan [2] to the Cauchy problem,

$$
\begin{aligned}
u_{xy} &= f(x,y,u,u_x,u_y) \ , \\
u_x &= \sigma(x) && \text{on } y = x, \\
u_y &= \tau(y) && \text{on } y = x, \\
u(0,0) &= 0 \ ,
\end{aligned}
$$

(1.2)

and to the mixed boundary value problem, i.e., boundary conditions in (1.2) replaced by

$$
u(0,y) = \tau(y) \ , \quad u(x,x) = \sigma(x) \ , \quad \sigma(0) = \tau(0) \ .
$$

In this paper we consider the 2-dimensional problem

$$
\begin{aligned}
u_{xy} &= f(x,y,u,u_x,u_y) \ , \\
u_x &= \sigma + au_y + bu && \text{on } y = \eta(x) \ , \\
u_y &= \tau + cu_x + du && \text{on } x = \zeta(y) \ , \\
\eta(0) &= \zeta(0) = 0 \ , \\
u(0,0) &= u_o, \text{ a constant} \ ,
\end{aligned}
$$

(1.3)

where σ, a, b are continuous functions of x, and τ, c, d are continuous functions of y. The method of proof used here is again by an extension of the method used in [3]. Using a different method, A. K. Aziz and J.B.Diaz[1]

have treated the corresponding problem for the linear equation $u_{xy} + au_x + bu_y + cu = d$ where a,b,c, and d are functions only of x and y. We refer the reader to [1] for a discussion of the history of the problem, and for an extensive bibliography. In the linear case Aziz and Diaz were able to obtain an existence theorem in the large. However for problem (1.3) all we get is an existence theorem in the small.

We note that the characteristic boundary value problem, the Cauchy problem, and the mixed boundary value problem are special cases of (1.3). So also is the Goursat problem, i.e., boundary conditions in (1.3) replaced by

$$
\begin{aligned}
u(x,\eta(x)) &= \sigma(x) \quad, \\
(1.4) \qquad u(\xi(y),y) &= \tau(y) \quad, \\
u(0,0) &= \sigma(0) = \tau(0) = 0 \quad.
\end{aligned}
$$

To see this differentiate the first two of (1.4), obtaining

$$
\begin{aligned}
u_x(x,\eta(x)) + \eta'(x)u_y(x,\eta(x)) &= \sigma'(x) \quad, \\
\xi'(y)u_x(\xi(y),y) + u_y(\xi(y),y) &= \tau'(y) \quad,
\end{aligned}
$$

and these are of the same form as the boundary conditions of (1.3).

2.   <u>The boundary value problem.</u>

Let R consist of the points (x,y) such that $0 \leq x \leq A$ and $0 \leq y \leq B$. Let S consist of those points (u,p,q) such that $-\infty < u,p,q < \infty$. And let C be the cartesian product R×S. The precise formulation of the problem to be solved is given in the following theorem:

Theorem 2.1. Hypotheses:

(a) f is a continuous real valued function of (x,y,u,p,q) in C.

(b) There is a constant $M > 0$ such that $|f(x,y,u,p,q)| < M$ in C.

(c) f satisfies a uniform Lipschitz condition in (p,q), i.e., there is a constant $L > 0$ such that

$$|f(x,y,u,\bar{p},\bar{q}) - f(x,y,u,p,q)| \leq L(|\bar{p}-p| + |\bar{q}-q|)$$

whenever the arguments of f are in C.

(d) σ, a, b, η are continuous functions of x for $0 \leq x \leq A$, and τ, c, d, ξ are continuous functions of y for $0 \leq y \leq B$.

(e) $|a(x) \cdot c(y)| < 1$ for (x,y) in R.

(f) $\xi(0) = \eta(0) = 0$, and $0 \leq \xi(y) \leq A$, $0 \leq \eta(x) \leq B$.

(g) There is a continuous function of x, say ζ, such that $\zeta(0) = 0$, $\zeta(A) = B$, and such that $y = \zeta(x)$ has an inverse for $0 \leq x \leq A$. Moreover the graph of $y = \eta(x)$ lies below that of $y = \zeta(x)$ and the graph of $x = \xi(y)$ lies above it, i.e., $\eta(x) \leq \zeta(x)$ and $\xi(y) \leq \zeta^{-1}(y)$.

(h) Letting $||a|| = \sup \left\{ |a(x)|: \ 0 \leq x \leq A \right\}$, etc., we assume that

$$A\left\{ \frac{||a||}{1 - ||a|| \ ||c||} \ \frac{||d|| + ||b||}{} \right\} + B\left\{ \frac{||b||}{1 - ||a||} \ \frac{||c|| + ||d||}{||c||} \right\} < 1 .$$

(i) $||a|| \ ||c|| \exp L(A+B) < 1$.

Conclusion:

There is at least one real valued function $u(x,y)$ which is continuous together with its partial derivatives u_x, u_y, u_{xy} in R, and which satisfies the partial differential equation

$$u_{xy}(x,y) = f[x,y,u(x,y), \ u_x(x,y), \ u_y(x,y)]$$

and which satisfies the following boundary conditions (here we use the notation $u_1(x,y) = u_x(x,y)$ and $u_2(x,y) = u_y(x,y)$),

$$u_1[x,\eta(x)] = \sigma(x) + a(x) u_2[x,\eta(x)] + b(x)u[x,\eta(x)] ,$$

$$u_2[\xi(y),y] = \tau(y) + c(y) u_1[\xi(y),y] + d(y)u[\xi(y),y] ,$$

$$u(0,0) = u_o .$$

The proof of this theorem will occupy the next several sections.

We note that the above boundary value problem is equivalent to the following system of integral equations, where $F(x,y) = f[x,y,u(x,y), u_1(x,y), u_2(x,y)]$.

$$u(x,y) = \int_0^x \left\{ \sigma(s)+a(s)u_2[s,\eta(s)]+b(s)u[s,\eta(s)] \right\} ds$$

$$+ \int_0^y \left\{ \tau(t)+c(t)u_1[\xi(t),t] + d(t)u[\xi(t),t] \right\}dt$$

(2.1)
$$+ \left\{ \int_0^y dt \int_{\xi(t)}^x ds - \int_0^x ds \int_0^{\eta(s)} dt \right\} F(s,t) + u_o ,$$

$$u_1(x,y) = \sigma(x)+a(x)u_2[x,\eta(x)] + b(x)u[x,\eta(x)] + \int_{\eta(x)}^y F(x,t)dt ,$$

$$u_2(x,y) = \tau(y)+c(y)u_1[\xi(y),y] + d(y)u[\xi(y),y] + \int_{\xi(y)}^x F(s,y)ds .$$

3. The finite difference scheme.

Let n be a positive integer and let $0 = x_o < x_1 < \ldots < x_n = A$. Let $y_j = \zeta(x_j)$ for $j = 0,\ldots,n$, where ζ is given in condition (g) of theorem 2.1. Then for $0 \leq j,k < n-1$ let

$$R_{j,k} = \left\{ (x,y): x_j \leq x < x_{j+1}, y_k \leq y < y_{k+1} \right\} ,$$

and

$$R_{n-1,k} = \left\{ (x,y): \ x_{n-1} \leq x \leq x_n \ , \ y_k \leq y < y_{k+1} \right\} \ ,$$

etc.

We next introduce the following functions. For (x,y) in $R_{j,k}$ let

$$\lambda(x) = \begin{cases} x_{j-1} & \text{if } j > 0 \\ x_j & \text{if } j = 0 \end{cases} \ , \qquad \mu(y) = \begin{cases} y_{k-1} & \text{if } k > 0 \\ y_k & \text{if } k = 0 \end{cases} \ ,$$

$$\hat{\pi}(x,y) = \sigma(x) + a(x)\rho(x,y) + b(x)\,\Theta(x,y) \ ,$$

$$\hat{\rho}(x,y) = \tau(y) + c(y)\pi(x,y) + d(y)\,\Theta(x,y) \ ,$$

$$\Theta(x,y) = \int_0^x \hat{\pi}[\lambda(s),\eta(\lambda(s))]ds + \int_0^y \hat{\rho}[\xi(\mu(t))], \mu(t)]dt$$

$$+ \left\{ \int_0^y dt \int_{\xi[\mu(t)]}^x ds - \int_0^x ds \int_0^{\eta[\lambda(s)]} dt \right\} \phi(s,t) + u_0 \ ,$$

$$\pi(x,y) = \begin{cases} (\Theta_{j+1,k} - \Theta_{j,k}) / (x_{j+1} - x_j), & j \geq 0, \ k \geq 1 \ , \\ \\ (\Theta_{j,0} - \Theta_{j-1,0}) / (x_j - x_{j-1}) \ , & j \geq 1, \ k = 0, \end{cases}$$

where $\Theta_{j,k} = \Theta(x_j, y_k)$, etc.,

$$\rho(x,y) = \begin{cases} (\Theta_{j,k+1} - \Theta_{j,k}) / (y_{k+1} - y_k), & j \geq 1, \ k \geq 0 \ , \\ \\ (\Theta_{0,k} - \Theta_{0,k-1}) / (y_k - y_{k-1}), & j = 0, \ k \geq 1 \ . \end{cases}$$

For (x,y) in $R_{0,0}$ let $\pi(x,y) = u_1(0,0)$, $\rho(x,y) = u_2(0,0)$ where

$$u_1(0,0) = \sigma(0) + a(0)\, u_2(0,0) \ ,$$

$$u_2(0,0) = \tau(0) + c(0)\, u_1(0,0) \ .$$

Let

$$\phi(x,y) = f[\lambda(x), \ \mu(y), \ \Theta(\lambda(x), \ \mu(y)), \ \pi(x,y), \ \rho(x,y)] \ .$$

One can now verify that if Θ, π, ρ have been computed in cells $R_{i,j}$ for $0 \leq i$, $j \leq k-1$, then it is possible to compute Θ, π, ρ in cells $R_{k,j}$ and $R_{i,k}$ for $0 \leq i$, $j \leq k$ in terms of quantities already computed in the previous cells.

If $j \geq 0$, $k \geq 1$ then for (x,y) in $R_{j,k}$

$$(3.1) \qquad \pi(x,y) = \pi_{j,k} = (\Theta_{j+1,k} - \Theta_{j,k}) / (x_{j+1} - x_j)$$

$$= \frac{1}{x_{j+1} - x_j} \left\{ \int_{x_j}^{x_{j+1}} \hat{\pi}[\lambda(s), \eta(\lambda(s))]ds \right.$$

$$\left. + \left[\int_0^{y_k} dt \int_{x_j}^{x_{j+1}} ds - \int_{x_j}^{x_{j+1}} ds \int_0^{\eta[\lambda(s)]} dt \right]\phi(s,t) \right\}$$

$$= \hat{\pi}[x_{j-1}, \eta(x_{j-1})] + \int_{\eta(x_{j-1})}^{y_k} \phi(x_j,t)dt ,$$

where x_{j-1} is to be replaced by x_j when $j = 0$. But

$$\frac{\partial}{\partial x} \Theta(x,y) = \hat{\pi}[\lambda(x), \eta(\lambda(x))] + \int_{\eta[\lambda(x)]}^{y} \phi(x,t)dt$$

$$= \hat{\pi}[x_{j-1}, \eta(x_{j-1})] + \int_{\eta(x_{j-1})}^{y} \phi(x,t)dt .$$

Since $\phi(x,y) = \phi(x_j,y)$, we have

$$\frac{\partial}{\partial x}\,\Theta(x,y_k) = \pi(x,y_k) \qquad \text{if } x_j < x < x_{j+1} \text{ ,}$$

$$\frac{\partial^+}{\partial x}\,\Theta(x_j,y_k) = \pi(x_j,y_k) \text{ .}$$

In the case of $j \geq 1$, $k = 0$ we get

$$\frac{\partial^+}{\partial x}\,\Theta(x_{j-1},y_k) = \pi(x_j,y_k) \text{ .}$$

Similarly if $j \geq 1$, $k \geq 0$ then

$$(3.2) \qquad \rho(x,y) = \rho(x_j,y_k) = (\Theta_{j,k+1} - \Theta_{j,k})\,/\,(y_{k+1} - y_k)$$

$$= \frac{1}{y_{k+1}-y_k}\left\{ \int_{y_k}^{y_{k+1}} \hat{\rho}[\xi(\mu(t)),\mu(t)]dt \right.$$

$$\left. + \int_{y_k}^{y_{k+1}} dt \int_{\xi[\mu(t)]}^{x_j} \phi(s,t)ds \right\}$$

$$= \hat{\rho}[\xi(y_{k-1}),y_{k-1}] + \int_{\xi(y_{k-1})}^{x_j} \phi(s,y_k)ds \text{ ,}$$

where y_{k-1} is to be replaced by y_k when $k = 0$. Then as above

$$\frac{\partial}{\partial y}\,\Theta(x_j,y) = \rho(x_j,y) \qquad \text{if } y_k < y < y_{k+1} \text{ ,}$$

$$\frac{\partial^+}{\partial y}\,\Theta(x_j,y_k) = \rho(x_j,y_k) \text{ .}$$

If $j = 0$, $k \geq 1$ then

$$\frac{\partial^+}{\partial y} \Theta(x_j, y_{k-1}) = \rho(x_j, y_k) .$$

We now introduce the functions p,q defined as follows:

$$p(x,y) = \begin{cases} \dfrac{\partial^+}{\partial x} \Theta(x,y), & x = x_j \\[3ex] \dfrac{\partial}{\partial x} \Theta(x,y), & x_j < x < x_{j+1} \end{cases}$$

$$q(x,y) = \begin{cases} \dfrac{\partial^+}{\partial y} \Theta(x,y), & y = y_k \\[3ex] \dfrac{\partial}{\partial y} \Theta(x,y), & y_k < y < y_{k+1} . \end{cases}$$

In what follows we will be interested in a sequence of subdivisions of R such that as n tends to infinity

$$(3.3) \qquad \alpha_n = \sup\left\{ |x_{j+1} - x_j| + |y_{k+1} - y_k| : \quad 0 \leq j < n, \; 0 \leq k < n \right\}$$

tends to zero.

To each n will correspond a particular Θ, π, ρ, p, q etc., and when we wish to indicate this explicitly we will write $\Theta^{(n)}, \pi^{(n)}, \rho^{(n)}, p^{(n)}, q^{(n)}$ etc.

4. The sequences of approximating functions.

We now want to show that the sequences $\left\{ \Theta^{(n)} \right\}, \left\{ p^{(n)} \right\}, \left\{ q^{(n)} \right\}$ contain subsequences $\left\{ \Theta^{(k(n))} \right\}, \left\{ p^{(k(n))} \right\}, \left\{ q^{(k(n))} \right\}$ such that $\Theta^{(k(n))} \to u$, $p^{(k(n))} \to u_1$, and $q^{(k(n))} \to u_2$, where u is a solution to the boundary value problem. Let us introduce the following definition.

9

Definition 1. A family of functions $\left\{ g^{(n)}(x,y) \right\}$ is equioscillating to zero on R if for any $\varepsilon > 0$ there is a $\delta > 0$ and an $n_0 > 0$ such that $\left| g^{(n)}(\bar{x},\bar{y}) - g^{(n)}(x,y) \right| < \varepsilon$ whenever $n > n_0$, $|\bar{x}-x| < \delta, |\bar{y}-y| < \delta, (\bar{x},\bar{y})$ in R and (x,y) in R.

We want to show that $\left\{ \Theta^{(n)} \right\}$, $\left\{ p^{(n)} \right\}$, $\left\{ q^{(n)} \right\}$ are uniformly bounded in absolute value and are equioscillating to zero on R.

Theorem 4.1. $\left\{ \Theta^{(n)} \right\}$, $\left\{ p^{(n)} \right\}$, $\left\{ q^{(n)} \right\}$ are uniformly bounded in absolute value on R.

Proof: We will first prove the theorem for $\left\{ \Theta^{(n)} \right\}$, $\left\{ \pi^{(n)} \right\}$, $\left\{ \rho^{(n)} \right\}$. From the definitions of Θ, π, ρ (dropping the superscript n) we see that there is a constant $M_1 > 0$ such that

$$||\pi|| \leq M_1 + ||a||\,||\rho|| + ||b||\,||\Theta|| ,$$
$$||\rho|| \leq M_1 + ||c||\,||\pi|| + ||d||\,||\Theta|| ,$$
$$||\Theta|| \leq M_1 + A\,||\pi|| + B\,||\rho|| .$$

Hence

$$||\pi|| \leq M_1 + ||a||\left\{ M_1 + ||c||\,||\pi|| + ||d||\,||\Theta|| \right\} + ||b||\,||\Theta|| ,$$

and since $||a||\,||c|| < 1$,

$$||\pi|| \leq \left\{ M_1(1+||a||) + (||a||\,||d|| + ||b||)||\Theta|| \right\} \left\{ 1 - ||a||\,||c|| \right\}^{-1}.$$

Similarly

$$||\rho|| \leq \left\{ M_1(1+||c||) + (||b||\,||c|| + ||d||)||\Theta|| \right\} \left\{ 1 - ||a||\,||c|| \right\}^{-1}.$$

Therefore,

$$||\Theta|| \leq M_1\left\{ 1 + \frac{A(1+||a||)+B(1+||c||)}{1 - ||a||\,||c||} \right\} +$$

$$+ \left\{ A(||a||\,||d||+||b||)+B(||b||\,||c||+||d||) \right\} \frac{||\Theta||}{1 - ||a||\,||c||}$$

and so by condition (h) of theorem (2.1) the $||\Theta||$'s are uniformly bounded, and hence so are the $||\pi||$'s and the $||\rho||$'s.

But it is obvious that the uniform boundedness of the $||\pi||$'s implies the uniform boundedness of the $||p||$'s, and similarly for the $||\rho||$'s and $||q||$'s.

In order to show that the sequences are equioscillating to zero we need the following result.

Theorem 4.2. If $0 \leq g(x) \leq M$ and $0 \leq g(x) \leq \gamma + L\left| \int_a^x g(s)ds \right|$ for $0 \leq a \leq A$, $0 \leq x \leq A$, then $g(x) \leq \gamma \exp(LA)$.

Proof: If $a < x$ then

$$g(x) \leq \gamma + L \int_a^x \left\{ \gamma + L \int_a^s g(s_1)ds_1 \right\} ds \leq \gamma(1+LA) + L^2 \int_a^x ds \int_a^s g(s_1)ds_1 ,$$

$$\leq \cdots \leq \gamma \sum_{n=0}^{N} \frac{(LA)^n}{n!} + M \frac{(LA)^{N+1}}{(N+1)!} .$$

Theorem 4.3. Each of the sequences $\left\{ \Theta^{(n)} \right\}$, $\left\{ p^{(n)} \right\}$, $\left\{ q^{(n)} \right\}$ is equioscillating to zero on R.

Proof: The theorem is obviously true for $\left\{ \Theta^{(n)} \right\}$. We next consider $\left\{ \pi^{(n)} \right\}$ and $\left\{ \rho^{(n)} \right\}$. Let us assume (x,y) in $R_{j,k}$ and $(\bar{x},\bar{y})$ in $R_{J,K}$ and j,J,k,K all ≥ 1. Again suppressing the superscript n, we have

$$\left| \pi(\bar{x},\bar{y}) - \pi(x,y) \right| \leq \left| \pi(\bar{x},\bar{y}) - \pi(\bar{x},y) \right| + \left| \pi(\bar{x},y) - \pi(x,y) \right| .$$

It is obvious that given $\varepsilon > 0$, there is a $\delta > 0$ such that (for n large enough) $\left| \pi(\bar{x},\bar{y}) - \pi(\bar{x},y) \right| < \varepsilon/2$ whenever $(\bar{y}-y) < \delta$. Hence we need only consider the term $\left| \pi(\bar{x},y) - \pi(x,y) \right|$. Similarly it is only necessary to consider $\left| \rho(x,\bar{y}) - \rho(x,y) \right|$. But then by (3.1), (3.2), and applying the Lipschitz condition to f,

$$|\pi(\bar{x},y) - \pi(x,y)| \leq \gamma + ||a|| \quad |\rho(x,\eta(x_j)) - \rho(x,\eta(x_j))|$$

$$+ L| \int_{\eta(x)}^{y} |\pi(\bar{x},t) - \pi(x,t)|dt|,$$

$$|\rho(x,\bar{y}) - \rho(x,y)| \leq \gamma + ||c|| \quad |\pi(\xi(y_K),y) - \pi(\xi(y_k),y)|$$

$$+ L \quad | \int_{\xi(y)}^{x} |\rho(s,\bar{y}) - \rho(s,y)|ds| ,$$

where $\gamma \geq 0$ is equioscillating to zero.

Therefore by theorem 4.2,

$$|\pi(\bar{x},y) - \pi(x,y)| \leq \left\{ \gamma + ||a|| \quad |\rho(x,\eta(x_j)) - \rho(x,\eta(x_j))| \right\} \exp(LA)$$

$$|\rho(x,\bar{y}) - \rho(x,y)| \leq \left\{ \gamma + ||c|| \quad |\pi(\xi(y_K),y) - \pi(\xi(y_k),y)| \right\} \exp(LB)$$

If we choose constants β, ω, ψ such that

$$0 \leq \gamma \exp(LA) = \beta, \quad 0 \leq \gamma \exp(LB) = \beta, \quad ||a||\exp(LA) = \omega, \quad ||c||\exp(LB) = \psi,$$

then

$$|\pi(x_J,y_k) - \pi(x_j,y_k)| \leq \beta + \omega|\rho[x_j,\eta(x_J)] - \rho[x_j,\eta(x_j)]| ,$$

$$|\rho(x_j,y_K) - \rho(x_j,y_k)| \leq \beta + \psi|\pi[\xi(y_K),y_k] - \pi[\xi(y_k),y_k]| .$$

Repeated application of these last inequalities gives

$$|\pi(x_J,y_k) - \pi(x_j,y_k)| \leq \beta + \omega\left\{\beta + \psi|\pi[\xi(\eta(x_J)),\eta(x_j)] - \pi[\xi(\eta(x_j)),\eta(x_j)]|\right.$$

$$= \beta(1+\omega) + \omega\psi|\pi[\xi(\eta(x_J)),\eta(x_j)] - \pi[\xi(\eta(x_j)),\eta(x_j)]| \leq \beta(1+\omega) \sum_{r=0}^{m} (\omega\psi)^r +$$

$$(\omega\psi)^{m+1}\sup\left\{ |\pi(\bar{x},y) - \pi(x,y)|; (\bar{x},y)\epsilon R, (x,y)\epsilon R \right\}.$$

Thus if $\omega\psi < 1$ then $\left\{ \pi^{(n)} \right\}$ is equioscillating to zero. But

12

$\omega\psi = ||a|| \ ||c|| \ \exp[L(A+B)]$, and this is < 1 by hypotheses (i) of theorem 2.1. We dispose of $\left\{ \rho^{(n)} \right\}$ in a similar way.

It is obvious from the definition of p,q that since $\left\{ \pi^{(n)} \right\}$, $\left\{ \rho^{(n)} \right\}$ are equioscillating to zero on R, then so are $\left\{ p^{(n)} \right\}$, $\left\{ q^{(n)} \right\}$.

5. Convergence to a solution.

Since each of the sequences $\left\{ \Theta^{(n)} \right\}$, $\left\{ p^{(n)} \right\}$, $\left\{ q^{(n)} \right\}$ is equibounded and equioscillating to zero, we can apply Arzela's theorem. More specifically we can find a subsequence $\left\{ n(s) \right\}_{s=1}^{\infty}$ of $\left\{ n \right\}_{n=1}^{\infty}$ such that $\Theta^{(n(s))}$ converges uniformly to a continuous limit function Θ^* on R. Then we can find a subsequence $\left\{ n(s(t)) \right\}_{t=1}^{\infty}$ of $\left\{ n(s) \right\}$ such that $p^{(n(s(t)))}$ converges uniformly to a continuous limit function p* on R. But then we can find a subsequence $\left\{ n(s(t(k))) \right\}_{k=1}^{\infty}$ of $\left\{ n(s(t)) \right\}$ such that $q^{(n(s(t(k))))}$ converges uniformly to a continuous limit function q* on R. If we now choose the original subsequence $\left\{ n(s) \right\}_{s=1}^{\infty}$ to coincide with $\left\{ n(s(t(k))) \right\}_{k=1}^{\infty}$ then for (x,y) in R

$$\lim_{s \to \infty} \Theta^{(n(s))}(x,y) = \Theta^*(x,y) \ ,$$

$$\lim_{s \to \infty} p^{(n(s))}(x,y) = p^*(x,y) \ ,$$

$$\lim_{s \to \infty} q^{(n(s))}(x,y) = q^*(x,y) \ .$$

We want to show that $\Theta^*(x,y)$ is a solution of the boundary value problem, and that

$$\frac{\partial \Theta^*}{\partial x} = p^* \quad , \quad \frac{\partial \Theta^*}{\partial y} = q^* \ .$$

13

To this end consider

$$(5.1) \quad \Theta^{(n(j))}(x,y) - \left\{ \int_0^x [\sigma(s) + a(s)q^*(s,\eta(s)) + b(s)\Theta^*(s,\eta(s))]ds \right.$$

$$+ \int_0^y [\tau(t) + c(t)p^*(\xi(t),t) + d(t)\Theta^*(\xi(t),t)]dt$$

$$\left. + [\int_0^y dt \int_{\xi(t)}^x ds - \int_0^x ds \int_0^{\eta(s)} dt]f[s,t,\Theta^*(s,t),p^*(s,t),q^*(s,t)] + u_0 \right\}.$$

Suppressing the superscript on $\Theta^{(n(j))}$ and using the fact that

$$p[\lambda(x),\mu(y)] = \pi(x,y), \quad q[\lambda(x),\mu(y)] = \rho(x,y),$$

the above expression is just equal to

$$\int_0^x \left\{ \sigma(\lambda(s)) - \sigma(s) + a(\lambda(s))q[\lambda(s),\eta(\lambda(s))] - a(s)q^*(s,\eta(s)) \right.$$

$$\left. + b(\lambda(s)) \Theta[\lambda(s),\eta(\lambda(s))] - b(s)\Theta^*(s,\eta(s)) \right\} ds$$

$$+ \int_0^y \left\{ \tau(\mu(t)) - \tau(t) + c(\mu(t))p[\xi(\mu(t)),\mu(t)] - c(t)p^*(\xi(t),t) \right.$$

$$\left. + d(\mu(t))\Theta[\xi(\mu(t)),\mu(t))] - d(t)\Theta^*(\xi(t),t) \right\} dt$$

$$+ \left\{ \int_0^y dt \int_{\xi(t)}^x ds - \int_0^x ds \int_0^{\eta(s)} dt \right\} \left\{ f[\lambda(s),\mu(t), \Theta(\lambda(s),\mu(t)), \right.$$

$$\left. p(\lambda(s),\mu(t)),q(\lambda(s),\mu(t))] - f[s,t,\Theta^*(s,t),p^*(s,t),q^*(s,t)] \right\} + \gamma_1$$

where $|\gamma_1| \le M\left\{ B \sup_{0 \le t \le B} |\xi[\mu(t)] - \xi(t)| + A \sup_{0 \le s \le A} |\eta[\lambda(s)] - \eta(s)| \right\}$,

and this last term goes to zero as the a_n (defined by (3.3)) goes to zero.

The absolute value of the integrand in the double integrals is not greater than

$$|f[\lambda,\mu,\Theta(\lambda,\mu),p(\lambda,\mu),q(\lambda,\mu)] - f[\lambda,\mu,\Theta^*(\lambda,\mu),p^*(\lambda,\mu),q^*(\lambda,\mu)]|$$

$$+ |f[\lambda,\mu,\Theta^*(\lambda,\mu),p^*(\lambda,\mu),q^*(\lambda,\mu)] - f[s,t,\Theta^*(s,t),p^*(s,t),q^*(s,t)]| \ .$$

By the uniform convergence of Θ,p,q to Θ^*,p^*,q^* respectively, and by the continuity of f,σ,τ,a,b,c,d it is clear that (5.1) goes to zero as $n(j) \to \infty$.

All that remains to do is to show that

$$\frac{\partial}{\partial x}\Theta^* = p^*, \qquad \frac{\partial}{\partial y}\Theta^* = q^* \ .$$

To do this proceed with p and q in the same way as we did with Θ, i.e., consider for example

$$p^{(n(j))}(x,y) - \left\{ \sigma(x) + a(x)q^*(x,\eta(x)) + b(x)\Theta^*(x,\eta(x)) \right.$$

$$\left. + \int_{\eta(x)}^{y} f[x,t,\Theta^*(x,t),p^*(x,t),q^*(x,t)]dt \right\} \ .$$

As in the case of (5.1) this expression goes to zero as $n(j) \to \infty$. Hence

$$\lim_{j \to \infty} p^{(n(j))} = p^* = \frac{\partial\Theta^*}{\partial x} \ .$$

Similarly we can show that $q^* = \frac{\partial\Theta^*}{\partial y}$. This completes the proof of theorem 2.1.

6. Other conditions on f.

It was shown by Glick [3] that in the case of the n-dimensional characteristic boundary value problem the condition that $|f|$ be bounded can be replaced by the condition that f satisfy a Lipschitz condition in u as well

as in the partial derivatives of u. The following theorem shows that we can get a similar result for problem (1.3).

Theorem 6.1. We make the same assumptions in theorem 2.1, except that conditions (b) and (c) are replaced by the following condition:

f satisfies a uniform Lipschitz condition in (u,p,q) i.e., there is a constant L > 0 such that $|f(x,y,\bar{u},\bar{p},\bar{q}) - f(x,y,u,p,q)| \leq L(|\bar{u}-u| + |\bar{p}-p| + |\bar{q}-q|)$ whenever the arguments of f are in C.

Then for A,B sufficiently small, we have the same conclusion as in theorem 2.1.

Proof: Let $\bar{f}(x,y,u,p,q) = f(x,y,0,0,0)$ and let $\bar{\theta}$, $\bar{\pi}$, $\bar{\rho}$ denote θ, π, ρ corresponding to the function $\bar{f}$. Then since $|\bar{f}|$ is bounded on R and hence on C, the sequences $\left\{ \bar{\theta}^{(n)} \right\}$, $\left\{ \bar{\pi}^{(n)} \right\}$, $\left\{ \bar{\rho}^{(n)} \right\}$ are uniformly bounded on R. Hence there is a constant M > 0 such that

$$||\pi|| \leq ||\bar{\pi}|| + ||\pi - \bar{\pi}|| \leq ||\bar{\pi}|| + ||a||\ ||\rho - \bar{\rho}|| + ||b||\ ||\theta - \bar{\theta}||$$

$$+ L\ |\ \int_{\eta}^{y} \left\{ ||\theta|| + ||\pi|| + ||\rho|| \right\} dt\ |$$

$$\leq M + ||a||\ ||\rho|| + ||b||\ ||\theta|| + LB(||\theta|| + ||\pi|| + ||\rho||).$$

Similarly

$$||\rho|| \leq M + ||c||\ ||\pi|| + ||d||\ ||\theta|| + LA(||\theta|| + ||\pi|| + ||\rho||)\ ,$$

and

$$||\theta|| \leq M + A(||a||\ ||\rho|| + ||b||\ ||\theta||) + B(||c||\ ||\pi|| + ||d||\ ||\theta||)$$

$$+ LAB(||\theta|| + ||\pi|| + ||\rho||)\ .$$

It is obvious that for A,B sufficiently small we can proceed as in section 4 to prove the uniform boundedness of $\Theta^{(n)}$, $\pi^{(n)}$, $\rho^{(n)}$. The rest of the proof goes through as before.

Finally we note that the boundedness of $|f|$ or the Lipschitz continuity of f with respect to u can be replaced by the condition

$$|f| < K(|u| + |p| + |q|)$$

for some constant $K > 0$. To see this it is only necessary to point out that in this case we have

$$||\pi|| \le ||\sigma||+||a||\ ||\rho||+||b||\ ||\Theta||+KB(||\Theta|| + ||\pi|| + ||\rho||) ,$$

$$||\rho|| \le ||\pi||+||c||\ ||\pi||+||d||\ ||\Theta||+KA(||\Theta|| + ||\pi|| + ||\rho||) ,$$

$$||\Theta|| \le A(||\sigma||+||a||\ ||\rho||+||b||\ ||\Theta||)+B(||\tau||+||c||\ ||\pi||+||d||\ ||\Theta||)$$
$$+ KAB(||\Theta|| + ||\pi|| + ||\rho||) .$$

and again, for A,B sufficiently small it follows that $\left\{\Theta^{(n)}\right\}$, $\left\{\pi^{(n)}\right\}$, $\left\{\rho^{(n)}\right\}$ are uniformly bounded.

This kind of condition on f has been used by Z. Szmydt [5] who treats a similar problem by a fixed point technique.

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